

Upper bounds for the number of orbital topological types of planar polynomial vector fields “modulo limit cycles”

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Abstract

The paper deals with planar polynomial vector fields. We aim to estimate the number of orbital topological equivalence classes for the fields of degree n . An evident obstacle for this is the second part of Hilbert’s 16th problem. To circumvent this obstacle we introduce the notion of *equivalence modulo limit cycles*. This paper is the continuation of the author’s paper in [Mosc. Math. J. **1** (2001), no. 4] where the lower bound of the form 2^{cn^2} has been obtained. Here we obtain the upper bound of the same form. We also associate an equipped planar graph to every planar polynomial vector field, this graph is a complete invariant for orbital topological classification of such fields.

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1 Introduction

Let us consider a planar polynomial vector field:

$$v(x, y) = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

Recall that vector fields v_1 and v_2 are called *orbitally topologically equivalent* if there is a homomorphism φ of the phase space such that φ takes any trajectory of v_1 to a trajectory of v_2 and φ preserves the natural orientation of these trajectories (see [2] and [3]).

The phase space of a vector field below is the plane, or a closed subspace of the plane such that its boundary consists of limit cycles of the field (see Definition 4).

It follows from Bezout's theorem that a singular point set of the polynomial vector field v is either finite or contains an algebraic curve. In the first case this set consists of at most $(\deg v)^2$ points. In the second case the equation of the algebraic curve is given by

$$F(x, y) = 0,$$

where F is the greatest common divisor of P and Q . We are mainly interested in the first case. Notice that we can, in some sense, reduce the second case to the first one, dividing P and Q by F .

Definition 1. A planar polynomial vector field is called **-field* if it has only a finite number of singular points.

1.1 Equivalence modulo limit cycles

One of the questions in the second part of the 16th Hilbert Problem is the following: Is it true that for any n there is a constant $H(n)$ such that every polynomial vector field of degree at most n has at most $H(n)$ limit cycles? The answer is not known even for $n = 2$.

Our goal is to find some upper and lower bounds for the number of orbital topological equivalence classes of degree n planar polynomial vector fields. The Hilbert Problem is an obstacle to obtaining such an estimate because orbitally topologically equivalent vector fields have the same number of limit cycles (see Corollary 1). To avoid this difficulty we shall introduce the concept of *equivalence modulo limit cycles* (see Definition 3 below). It is possible to obtain both upper and lower bounds for this type of equivalence. The aim of this paper is to obtain the upper bound. The lower bound has been found in [5].

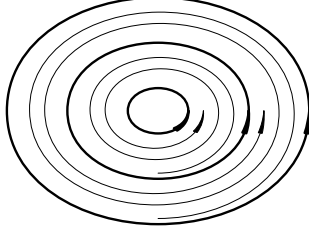


Figure 1: A nest.

Definition 2. A *nest* of a vector field v (see Figure 1) is an open subset Z of its phase space such that

1. Z is homeomorphic to an annulus;
2. the boundary curves of Z are limit cycles of the field;
3. Z contains no singular points of the field.

Definition 3. Let us consider vector fields v_1 and v_2 . Let Z_1 (Z_2) be the union of all the nests of v_1 (v_2). The fields v_1 and v_2 are called *equivalent modulo limit cycles* if the restriction of v_1 to $\mathbb{R}^2 \setminus Z_1$ is orbitally topologically equivalent to the restriction of v_2 to $\mathbb{R}^2 \setminus Z_2$.

1.2 Main Result

Theorem 1. Denote by $K(H, n)$ the number of orbital topological equivalence types of planar polynomial vector fields v such that 1) v has finite number of singular points; 2) v has at most H limit cycles and 3) $\deg v \leq n$. Then

$$K(H, n) \leq C^{H+n^2},$$

where $C = 10^{157}$.

Corollary 1. For every $n > 0$ the following statements are equivalent

1. The number of orbital topological equivalence classes of planar polynomial vector fields with finite number of singular points and degree less than or equal to n is finite.
2. There is $H(n)$ such that every planar polynomial vector field of degree less than or equal to n has at most $H(n)$ limit cycles.

Proof. $2 \implies 1$. This is obvious from Theorem 1.

$1 \implies 2$. We can restrict ourselves to $*$ -fields. Indeed, if we reduce components of a vector field by the common divisor, the number of limit cycles can only increase (see the remark before Definition 1).

Suppose that there are exactly m orbital topological classes of $*$ -fields of degree less than or equal to n . Take one representative v_i for each class ($i = 1, \dots, m$). Let H_i be the number of limit cycles of v_i . By the Finiteness Theorem (see [4], [6] and also [7]) $H_i < \infty$. Thus

$$H(n) = \max_{1 \leq i \leq m} H_i < \infty.$$

□

Theorem 2. *Consider planar polynomial vector fields with finite number of singular points and degree at most n . Denote by $M(n)$ the number of equivalence classes modulo limit cycles of such fields. Then*

$$c^{n^2} \leq M(n) \leq C^{n^2},$$

where $C = 10^{471}$, $c = 10^{10^{-8}}$.

Both Theorems above follow from a general Theorem below.

Definition 4. A closed set $\Pi \subset \mathbb{R}^2$ is called *admissible* for a $*$ -field v if its boundary is a union of some limit cycles of v . The restriction w of the $*$ -field to an admissible set Π is called a *P-field*. All the limit cycles of v that are components of $\partial\Pi$ are considered limit cycles of w . The *degree* of w is the degree of v .

Theorem 3. *Consider P-fields of degree at most n with at most H limit cycles. There are at most*

$$C^{H+n^2}$$

orbital topological equivalence classes of such fields, where $C = 10^{157}$.

Proof of Theorem 1. Notice that \mathbb{R}^2 is an admissible set for any $*$ -field. □

Proof of Theorem 2. The lower bound in the Theorem is proved in [5]. The explicit constant $c = 10^{10^{-8}}$ has not been written out but the calculation is straightforward.

Take pairwise non-equivalent modulo limit cycles $*$ -fields v_1, \dots, v_M with $\deg v_i \leq n$ for $i = 1, \dots, M$. Let D_i be the union of all the nests of v_i . The set $\mathbb{R}^2 \setminus D_i$ is admissible for v_i , since D_i is the disjoint union of all the maximal nests of v_i . Let w_i be the restriction of v_i to $\mathbb{R}^2 \setminus D_i$. By Definition 3 the fields w_i are pairwise orbitally topologically non-equivalent.

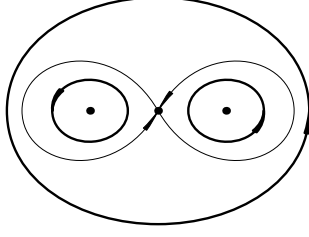


Figure 2: Invariant set of nonzero Euler characteristic must contain a singular point.

Lemma 1.1. *Sum of the number of maximal nests of a $*$ -field and the number of its limit cycles that do not belong to any nest is less than or equal to the number of singular points of this $*$ -field.*

It follows from this Lemma and from Bezout's Theorem that the number of limit cycles of w_i is at most $2n^2$ (note that every maximal nest gives two boundary limit cycles). Now we can apply Theorem 3:

$$M \leq C^{2n^2+n^2} = (C^3)^{n^2},$$

where $C = 10^{157}$. Hence, $M(n) \leq (C^3)^{n^2} = 10^{471(H+n^2)}$. \square

Proof of Lemma 1.1. Choose a limit cycle in every maximal nest. Call these limit cycles and the limit cycles that do not belong to the nests *labelled*. Our goal is to assign a singular point to every labelled limit cycle.

Take any labelled cycle L . Let L_1, \dots, L_s be all the labelled cycles, satisfying two conditions:

1. L_i is inside L .
2. There is no labelled cycle L' such that L' is inside L and L_i is inside L' .

Let D be the domain bounded by L , let D_i be the domain bounded by L_i . Consider

$$\Omega = D \setminus \overline{(D_1 \cup \dots \cup D_s)}.$$

We claim that Ω contains at least one singular point. Indeed, if $s > 1$, then it follows from the fact that Euler characteristic of Ω is not equal to 0 (see Figure 2). Suppose $s = 1$ and there are no singular points in Ω . Then Ω is a nest, so L and L_i are in the same maximal nest. We come to contradiction.

Assign to L any singular point inside Ω . Clearly, we have assigned different singular points to different labelled cycles. \square

The rest of the paper is devoted to the proof of Theorem 3. The paper is organized as follows: in Section 2 we recall the topological classification of singular points, define *complexity* of a singular point, and estimate the sum of complexities of all the singular points of a P -field, using the method, which we have learnt from [8]. In Section 3 we assign a graph on the sphere to a P -field. It is proved in Appendix, that this graph is a complete invariant of orbital topological classification of P -fields. The proof is essentially using Theorems 75 and 76 of [1] (see also [9] and [10]). In Section 4 we estimate the number of graphs on the sphere, using the main result of [11].

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2 Singular points

A P -field cannot be flat at any point. Recall (see [2], Chapter 5, §3) that every non-flat singular point O of a smooth planar vector field is either monodromic or has a characteristic trajectory (i.e. a trajectory that tends to O as $t \rightarrow +\infty$ or $t \rightarrow -\infty$, being tangent to some line at O).

2.1 Classification of monodromic singular points

Topological type of monodromic singular point is determined by its Poincare map. This map cannot have infinite number of isolated fixed points due to the Finiteness Theorem, since these fixed points correspond to limit cycles (see [4], [6] and also [7]). Thus every monodromic singular point of a P -field is either a focus, or a center.

2.2 Classification of characteristic singular points

A small neighbourhood of a characteristic singular point O splits into the union of standard sectors. There are three types of standard sectors: hyperbolic, elliptic, and parabolic (see [2], Chapter 5, §3 and Figure 3).

Remark 2.1. This splitting is not canonical: if we shrink the neighbourhood, then some parts of the elliptic sector will become parabolic sectors. Nevertheless Definition 5 and Definition 6 are “invariant”.

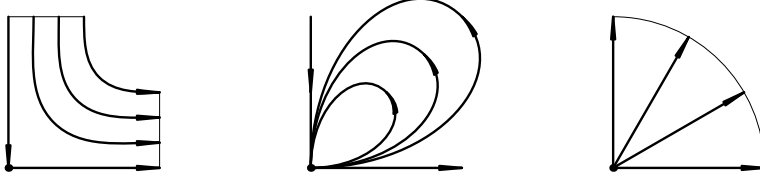


Figure 3: Hyperbolic, elliptic, and parabolic sectors.

2.3 Compactification of the phase space

We want to compactify the phase space by adding an infinite point to the plane. To get a chart in the neighbourhood of this point we identify the plane with the 2-sphere, punctured at the North Pole, by stereographic projection. The projection from the South Pole gives a chart in the neighbourhood of the infinite point. The transition between these charts is given by

$$(x, y) \mapsto \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Thus in the second chart our vector field has the form

$$\frac{v_1(x, y)}{(x^2 + y^2)^n},$$

where v_1 is a polynomial vector field. Set

$$v_\infty(x, y) = (x^2 + y^2)v_1(x, y). \quad (1)$$

The fields of lines, corresponding to v and v_∞ , agree on $\mathbb{R}^2 \setminus (0, 0)$. Thus they can be patched together to the field v_{dir} on the sphere.

The infinite point is a singular point of v_{dir} , thus the trajectories of v_{dir} are those of v plus the infinite point. This is why we need an extra $x^2 + y^2$ factor in (1) (otherwise we could have a trajectory of v_{dir} corresponding to two trajectories of v).

The following convention will be taken: the infinite point is considered a singular point of v .

2.4 Complexity of singular points

Definition 5. *Complexity* of a characteristic singular point is the number of its hyperbolic and elliptic sectors. Complexity of a monodromic singular point is zero.

We are going to estimate the total complexity (i.e. the sum of complexities) of singular points. First we estimate the total complexity of all the singular points except the point at infinity.

Proposition 2.1. *The total complexity of finite singular points of a degree n P -field is at most $6n^2 - 2n$.*

Proof. Consider a P -field of degree $n \geq 1$

$$v(x, y) = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

Set $R = P^2 + Q^2$. Then $R(x, y) = 0$ is the equation for the set of singular points of v .

Lemma 2.1. *Let Ω be any open bounded set of the plane, containing all the singular points of v . Let γ_ε be the curve given by the equation $R(x, y) = \varepsilon$. Then*

1. *For sufficiently small $\varepsilon > 0$ the curve γ_ε is nonsingular in Ω .*
2. *The ovals of this curve tend to singular points of v as $\varepsilon \rightarrow 0$.*
3. *For each singular point there is an oval, enveloping this point.*

Consider the set S of points on γ_ε where v is tangent to this curve. This set is given by the system of equations

$$\begin{cases} L_v(R - \varepsilon) = 0 \\ R - \varepsilon = 0, \end{cases}$$

where L_v is the directional derivative in the direction of v . These equations are polynomial of degree $3n - 1$ and $2n$ respectively. Thus S is algebraic. Let ε be so small that γ_ε is nonsingular. Clearly, if an oval of γ_ε contains infinitely many points of S , then this oval is contained in S . In this case the oval is a cycle of v . Thus S consists of isolated points and cycles of v .

Now Bezout's Theorem tells us that S has at most $6n^2 - 2n$ isolated points. It is an easy consequence of Lemma 2.1 that in every hyperbolic or elliptic sector of a singular point of v there is a point of γ_ε where v is tangent to this curve; for ε small enough these points are distinct and isolated (the latter follows from the fact that a characteristic singular point has a neighbourhood without cycles). Thus the overall number of elliptic and hyperbolic sectors is bounded by $6n^2 - 2n$. \square

Proof of Lemma 2.1. The singular set of the curve γ_ε is its intersection with the set, given by the equation $\text{grad } R = 0$. We claim that every singular point O has a punctured neighbourhood without points where $\text{grad } R = 0$. If not, then there is a smooth curve in this set such that O is in its closure, since every algebraic set is a union of finite number of smooth strata.

Since R does not change along this curve, the curve consists of singular points of v but this is impossible. Thus there is a punctured neighbourhood of O without points of the set $\text{grad } R = 0$, and 1) is proved.

Let O_1, \dots, O_k be all the singular points of v . Choose pairwise disjoint open neighbourhoods U_i such that $\overline{U_i} \subset \Omega$. Set

$$\delta = \min\{R(x, y) : (x, y) \in \overline{\Omega} \setminus \cup_{i=1}^k U_i\}.$$

Clearly $\delta > 0$. Take any $\varepsilon < \delta$. We have

$$\{(x, y) : R(x, y) < \varepsilon\} \cap \Omega \subset \cup_{i=1}^k U_i.$$

This proves 2).

It remains to show that in U_i there is an oval, enveloping O_i . But if there is no such an oval, then we can join O_i with some point of $\Omega \setminus \cup_{i=1}^k U_i$ by a curve that does not intersect γ_ε , this contradicts continuity of R . \square

Proposition 2.2. *The complexity of the infinite point is at most $2n + 2$.*

Proof. It is similar to the proof of Proposition 2.1. The only difference is that we should use the curve $x^2 + y^2 = C$ with C large enough instead of the curve $P^2 + Q^2 = \varepsilon$. \square

3 The structure of P -fields

We shall assign an equipped oriented graph on the sphere (possibly disconnected) to every P -field. This graph can have loops and multiple edges. This graph will be a complete invariant of the orbital topological type of a P -field. Two graphs are considered equivalent if they are isomorphic as equipped graphs, embedded into the sphere.

Definition 6. A *separatrix* of a P -field is a boundary trajectory of a hyperbolic sector of a characteristic singular point.

3.1 Small graph

We want to assign an edge of the graph to every separatrix of the P -field. The problem is that the α -limit set or the ω -limit set of the separatrix can

consist of more than one point. Also, we want to have the graph C^1 -smooth. Thus we first introduce the notion of *small graph*.

Definition 7. Let $a \in \mathbb{R}$ or $a = -\infty$. An α -germ at a is an equivalence class of maps $(a, b) \rightarrow S^2$, where two maps $(a, b_1) \rightarrow S^2$ and $(a, b_2) \rightarrow S^2$ are equivalent if their restrictions onto (a, b_3) coincide for some b_3 . The ω -germ is defined similarly.

An α -germ at $a \neq -\infty$ is called C^1 -smooth if some (and then any) of its representatives can be extended to a C^1 -smooth map $(a - \varepsilon, b) \rightarrow S^2$ for some $\varepsilon > 0$.

For an α -germ at $-\infty$ we consider a representative $\gamma : (-\infty, b) \rightarrow S^2$. The α -germ is C^1 -smooth if the map $\gamma \circ \tan : (-\frac{\pi}{2}, \tan^{-1} b) \rightarrow S^2$ can be extended to a C^1 -smooth map $(-\frac{\pi}{2} - \varepsilon, \tan^{-1} b) \rightarrow S^2$ for some $\varepsilon > 0$. The similar definitions apply to ω -germs.

Let γ be a trajectory of a P -field. Assume that γ is neither a singular point, nor a cycle, then it has a natural parametrization $\gamma : (a, b) \rightarrow S^2$ (possibly $a = -\infty$, $b = +\infty$). The α -germ of the trajectory is the α -germ of γ at a , the ω -germ is the ω -germ of γ at b .

Definition 8. A separatrix of a P -field is called *nice* if both its α -germ and ω -germ are C^1 -smooth. Other separatrices are called *nasty*.

Lemma 3.1. *The α -germ of a trajectory τ is C^1 -smooth provided this trajectory tends to a characteristic singular point O as $t \rightarrow -\infty$. The similar statement is valid for ω -germs.*

Proof. By Theorem 64 of §20 of [1], τ tends to O in a definite direction. Choose an affine coordinate system in the neighbourhood of O so that the direction is parallel to the x -axis. Suppose the α -germ of τ is not C^1 -smooth.

Step 1. We claim that there is a direction, not parallel to x -axis, such that the points on τ where the tangent line is parallel to this direction accumulate to O . Indeed, if in no neighbourhood of O the projection of τ to x -axis is one-to-one, then we can take the direction of y -axis. Otherwise in some neighbourhood of O the curve τ is given by an equation $y = g(x)$, $x > 0$. We have

$$\frac{g(x)}{x} \rightarrow 0, \text{ as } x \rightarrow 0^+, \quad (2)$$

since τ is tangent to x -axis. Since the α -germ of τ is not C^1 -smooth, $\lim_{x \rightarrow 0^+} g'(x) \neq 0$. Indeed, otherwise we can extend $g(x)$ to a C^1 -smooth function, declaring $g(x) = 0$ for $x \leq 0$. Now it follows from (2) and the Mean Value Theorem that $\lim_{x \rightarrow 0^+} g'(x)$ does not exist. Let λ be any nonzero number

between the lower and upper limits of $g'(x)$. We can take direction of the line $y = \lambda x$.

Step 2. The set of points in the phase space where the vector field is parallel to this direction is an algebraic set. Thus its intersection with small enough neighbourhood of O is the union of smooth curves $\gamma_1, \dots, \gamma_j$, where γ_i connects O with some point O_i . It is enough to show that for all i the intersection points of γ_i with τ cannot accumulate to O . This is clear if γ_i is not tangent to τ at O . Otherwise we shall use a version of Rolle–Khovanskii method.

Assume that the points of intersection of τ with γ_i accumulate to O . Since τ and γ_i are analytic these points can accumulate only to O . Thus we can enumerate them in the order they appear on τ :

$$M_1, M_2, \dots,$$

where $M_s \rightarrow O$ as $s \rightarrow \infty$. If the field is tangent to γ_i everywhere, then γ_i is part of a trajectory, and the claim is easy. Otherwise, shrinking the neighbourhood of O we can assume that the field is transversal to γ_i (except at O). Consider any curve γ'_i , satisfying the following properties: (1) It connects O with O_i ; (2) It does not intersect γ_i ; (3) It is tangent to y -axis at O . Then γ_i and γ'_i bound a domain, denote it by Ω . The part of τ between M_s and M_{s+1} cannot be entirely in Ω , since the field is transversal to γ_i . Thus there is a point of τ between M_s and M_{s+1} , where τ intersects γ'_i .

We see that if the intersection points of γ_i with τ accumulate to O , then so do the intersection points of γ'_i with τ . This contradicts property (3) of γ'_i . \square

Remark 3.1. The converse is also true but we do not need this.

Definition 9. Choose a point M_i on every limit cycle (including boundary limit cycles, see Definition 4). Choose one trajectory in every elliptic sector of every singular point. The *small graph* consists of the following elements:

Vertices: The points M_i and the singular points of the P -field.

Edges: The closures of the chosen trajectories in elliptic sectors, the limit cycles, and the closures of the nice separatrices.

This graph is naturally embedded into the sphere. All the edges of the small graph are C^1 -smooth in this embedding (for trajectories in elliptic sectors it follows from Lemma 3.1).

3.2 Limit polycycles

Consider a C^1 -smooth curve on the 2-sphere. By *side* of the curve we mean the choice of co-orientation of this curve. Thus every C^1 -smooth curve has 2 sides.

Definition 10. Let γ_1 and γ_2 be co-oriented separatrices of the vector field (i.e. one of the two possible co-orientations on each separatrix is chosen). We say that γ_2 is the *continuation* of γ_1 if the following holds:

1. γ_1 tends to a singular point O as $t \rightarrow +\infty$, γ_2 tends to O as $t \rightarrow -\infty$;
2. γ_1 and γ_2 bound a hyperbolic sector of O ; this sector is on the positive side of each of the separatrices.

Definition 11. A *limit polycycle* of a vector field is a cyclically ordered finite set of singular points (with possible repetitions), and a cyclically ordered set of disjoint co-oriented separatrices such that

1. the time oriented j -th separatrix connects the j -th and $(j + 1)$ -st singular points;
2. the $(j + 1)$ -st separatrix is the continuation of the j -th separatrix.

(compare with [7], §3.1.)

Consider a half-interval with the vertex on the limit polycycle (not at a singular point). Assume that this half-interval is transversal to the field everywhere and it is on the positive side of the polycycle. The monodromy map g is defined in some neighbourhood of the vertex. Let z be the coordinate on the half-interval ($z = 0$ at the vertex, $z > 0$ outside of the vertex).

The fixed points of g correspond to limit cycles, thus there are only finitely many of them by the Finiteness Theorem (see [4], [6] and also [7]). Thus for all z small enough we have $g(z) > z$, $g(z) = z$ or $g(z) < z$. In these cases we call the limit polycycle α -limit, 0-limit and ω -limit respectively.

Clearly, an α -limit (ω -limit) polycycle is the α -limit (ω -limit) set for all nearby trajectories that are on its positive side. The converse is also true.

Lemma 3.2. *The α -limit (ω -limit) set of any trajectory is either a point, or a limit cycle, or a limit polycycle.*

Proof. Consider any infinite limit set X such that X is not a limit cycle. By Theorem 68 of §23 of [1] the separatrices, constituting X , can be cyclically ordered so that for any two consecutive separatrices the next is the continuation of the previous. The continuation is defined differently in [1] (see Definitions 19 and 20 of §15). We leave to the reader to check that this definition matches our definition. It follows that X is a limit polycycle. \square

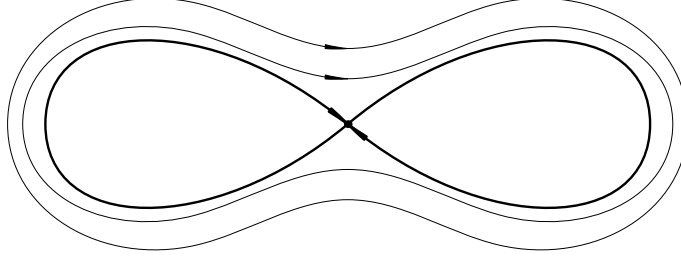


Figure 4: A 0-limit polycycle.

3.3 Cycles without contact and large graphs

Definition 12. A *cycle without contact* of a vector field is a smooth closed curve transversal to the vector field at every point.

It is easy to see (see [1], Lemma 2 of §24 and Lemma 1 of §27) that we can assign a cycle without contact to every α -limit polycycle, every ω -limit polycycle, every focus, and every side of a limit cycle so that

1. Consider the region bounded by a limit polycycle, a focus, or a limit cycle and the corresponding cycle without contact. This region contains neither singular points, nor limit cycles, nor other chosen cycles without contact.
2. Every trajectory, tending to a polycycle, to a focus, or to a limit cycle, intersects the corresponding cycle without contact at exactly one point. Every trajectory that intersects a cycle without contact tends to corresponding focus, polycycle, or a limit cycle.
3. These cycles without contact intersect neither each other nor the elements of the small graph.

Fix such a system of cycles without contact.

Lemma 3.3. *Every nasty separatrix intersects exactly one cycle without contact at exactly one point.*

Proof. We can assume that ω -limit set of a nasty separatrix γ is a characteristic limit point. It follows from Lemmas 3.1 and 3.2 that its α -limit set is a focus, a limit polycycle, or a limit cycle. Denote the corresponding cycle without contact by Y . Then γ intersects Y at exactly one point. It remains to prove that γ does not intersect other cycles without contact. Indeed, if

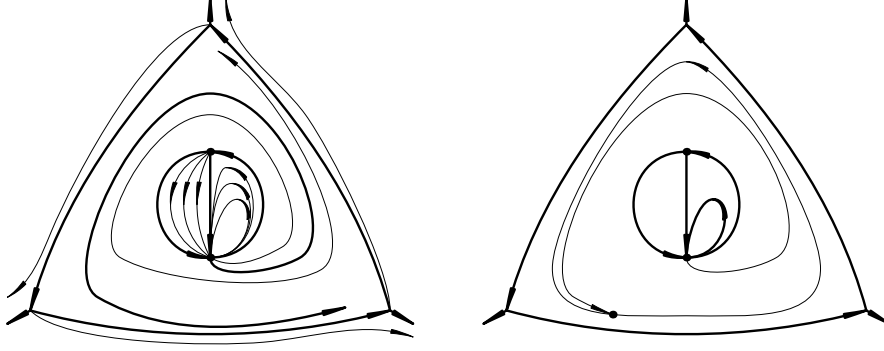


Figure 5: A phase portrait and the corresponding large graph (thick curves on the right picture correspond to the edges of the small graph).

γ intersects another cycle Y' , then the corresponding focus, limit cycle, or limit polycycle should coincide with α -limit set of γ .

Thus Y and Y' correspond to the same α -limit set. There are only two situations when it happens: (1) Y and Y' are the cycles without contact corresponding to the same limit cycle and (2) Y and Y' correspond to limit polycycles which coincide as sets but have different co-orientation of trajectories. The first case is impossible, since γ would have to intersect this limit cycle. In the second case the polycycle is necessarily homeomorphic to the circle; γ would have to intersect the polycycle, which is impossible. \square

Thus a nasty separatrix is cut by the point of its intersection with the corresponding cycle without contact into two parts. The part that is the boundary trajectory of a hyperbolic sector is called *truncated separatrix*.

Definition 13. The *large graph* has all the vertices and edges of the small graph and the following

Vertices: The intersection points of the nasty separatrices with the chosen cycles without contact.

Edges: The truncated nasty separatrices and the segments of the cycles without contact in which they are split by the vertices, oriented counterclockwise (if the cycle intersects no separatrix, then this cycle is not added to the large graph).

3.4 Equipping the graph

Now we want to equip vertices and edges of the graph with some data. We indicate on a vertex whether it is a repeller, an attractor, a clockwise

rotation center, a counterclockwise rotation center, or none of the above (e.g. a saddle). Thus there are **5** possible equipplings of a vertex.

On every edge we indicate whether it is a part of a cycle without contact, a trajectory of an elliptic sector, a separatrix, or an edge, corresponding to a limit cycle.

Consider a limit polycycle X . Its separatrix ψ has a distinguished side with respect to this polycycle. We mark this side of ψ by the symbol α , ω , or 0 , depending on the type of polycycle.

It the side of a separatrix is not marked with α , ω , or 0 (e.g. the separatrix does not belong to any limit polycycle) we mark this side with \emptyset . In the same way we equip the edge, corresponding to a limit cycle. However, only α , ω , and \emptyset are possible for a side of a limit cycle. Indeed, since the Poincare map of a limit cycle is analytic there are only two possibilities: this map is either identity, or the set of its fixed points is discrete. It cannot be identity, since the cycle is limit, thus 0 is impossible on a limit cycle. The combination $\emptyset\emptyset$ is also impossible. The other combinations are possible: for example, $\omega\emptyset$ means that the limit cycle is a stable boundary limit cycle (see Definition 4). Thus there are **8** possible equipplings of a limit cycle.

For a separatrix all the combinations are possible. If a separatrix marked with anything except $\emptyset\emptyset$, then it is nice and its α -germ and ω -germ are both boundary trajectories of some hyperbolic sectors by definition of limit polycycle.

If a separatrix is marked with $\emptyset\emptyset$, then we shall indicate if its α -germ and ω -germ are boundary trajectories of hyperbolic sectors. There are **3** cases because at least one germ should be a boundary trajectory of a hyperbolic sector by definition of separatrix. Notice that for a nasty separatrix the equippling is unnecessary but we shall keep it for simplicity.

Thus there are **15 + 3 = 18** possible equipplings of a separatrix. It adds up to **1 + 1 + 8 + 18 = 28** possible equipplings of an edge.

Proposition 3.1. *Suppose that two P -fields have their equipped graphs isomorphic (as planar oriented equipped graphs). Then these P -fields are orbitally topologically equivalent.*

The proof of this Proposition is a reformulation of the main result of [1] and is given in the Appendix.

4 Combinatorics

Proposition 4.1. *Let $G_{\text{conn}}(l)$ be the number of connected graphs on the sphere (possibly with loops and multiple edges) with at most l edges. Then*

$$G_{\text{conn}}(l) < 12^l. \quad (3)$$

(two graphs Γ_1 and Γ_2 are considered the same if they are isomorphic as embedded graphs.)

Proof. The graph is said to be *rooted* if one of its edges is chosen together with its orientation and co-orientation (the other edges are neither oriented, nor co-oriented). Let a_l be the number of rooted graphs with l edges. By (5.1) of [11]

$$a_l = \frac{2(2l)!3^l}{l!(l+2)!}.$$

A root is an additional structure. Thus if we forget the root, then we decrease the number of graphs:

$$G_{\text{conn}}(l) \leq a_1 + \dots + a_l \leq la_l = \frac{2l}{(l+2)(l+1)} 3^l \binom{2l}{l} < 12^l.$$

□

Remark 4.1. There is another approach due to physicists. Since every graph can be completed to a triangulation, and the number of subgraphs of a triangulation grows exponentially, it is enough to estimate the number of triangulations. The following integral is the generating function for these numbers (for graphs of different genera):

$$\int_{\mathcal{H}_N} e^{-\frac{t}{N} \text{tr}(H^3)} d\mu(H),$$

where \mathcal{H}_N is the set of $N \times N$ hermitian matrices, $d\mu$ is the Gaussian measure (see [12], §7.1). However, the author has never seen the proof of the required estimate, based on these methods.

Definition 14. The *size* of a graph is the sum of the number of its edges and the number of its vertices.

Proposition 4.2. *Let $G_{\text{dc}}(l)$ be the number of oriented possibly disconnected graphs on the sphere (possibly with loops and multiple edges) with size at most l . Then*

$$G_{\text{dc}}(l) < 48^l. \quad (4)$$

Proof. Consider a graph Γ with b edges and f vertices ($b + f \leq l$). Suppose that this graph consists of d connected components, then $d \leq f$. Clearly, we can make the graph connected by adding $d - 1$ edges. Denote the new graph by Γ' . Then the number of edges of Γ' is less than l .

Thus every planar graph of size at most l can be obtained from a *connected* graph with at most l edges by deleting some edges. Since there are at most 2^l subsets of edges of Γ' , there are at most $2^l G_{conn}(l)$ planar graphs with size l or less. Putting an orientation on every edge, gives 2^b factor. Thus

$$G_{dc}(l) \leq 2^b \cdot 2^l \cdot G_{conn}(l) < 48^l.$$

□

In Section 3 we have assigned an equipped graph to every P -field.

Proposition 4.3. *The size of the graph, corresponding to a P -field of degree n with H limit cycles, is at most $2H + 37n^2 + 13$.*

Proof. Denote by e and h the total numbers of elliptic and hyperbolic sectors respectively. Denote by s_1 and s_2 the number of nice and nasty separatrices respectively. Clearly, $s_1 + s_2 \leq 2h$. The number of vertices of the small graph is at most $H + (n^2 + 1)$ (remember the infinite singular point!), the number of its edges is $H + s_1 + e$.

The number of additional vertices of the large graph is s_2 , the number of additional edges is $2s_2$. Thus the size of the graph is at most

$$6h + e + 2H + n^2 + 1. \tag{5}$$

By Propositions 2.1 and 2.2 $e + h$ (this is the total complexity of singular points) is at most $6n^2 + 2$ ($2n + 2$ for the infinite point, $6n^2 - 2n$ for the finite points). It remains to substitute the last inequality in (5). □

Now we can finish the proof of Theorem 3. The Theorem trivially holds for $n = 0$, since all the constant P -fields are topologically equivalent. Thus we can assume $n > 0$. Set $N = 2H + 37n^2 + 13$. Using Propositions 4.2 and 4.3, we see that there are at most 48^N possible graphs. Every vertex of the graph can have one of 5 possible equipplings, every edge of the graph can have one of 28 possible equipplings, thus there are at most 28^N possible equipplings. Therefore there are at most $(28 \cdot 48)^N$ possible equipped graphs.

Now we apply Proposition 3.1, it gives the estimate $(28 \cdot 48)^N$ for the number of P -fields. Further,

$$(28 \cdot 48)^N \leq 1344^{50(H+n^2)} < (10^{157})^{H+n^2}.$$

Theorem 3 is proved.

A Appendix: Proof of Proposition 3.1

In [1] to every vector field on the sphere with finite number of “singular elements” a *scheme* is assigned. The main Theorems of [1] (Theorems 75 and 76 of §29) assert that if two vector fields have the same schemes, then they are orbitally topologically equivalent. Unfortunately, the definition of scheme (Definition 33, §29) is distributed all over the book.

So our goal is to show that the scheme of a P -field can be recovered from the large graph of the field. We do this examining subsequently all the elements of the scheme. According to Definition 33 of §29, we need to list all the *singular elements*, *limit continua*, their *global schemes*, and all the pairs of *conjugate free continua*. We shall recall all the relevant definitions from [1].

The other issue is that [1] deals with bounded phase spaces. However, they mention that all the results are valid for a system on the sphere, see §29.5. We shall have to adjust some definitions of [1] to this case.

By default all the references in this Appendix are the references to [1].

A.1 Singular elements

According to Definition 33, there are 8 types of singular elements:

1. Equilibrium states are singular points in our terminology. They are vertices of our graph. Notice that we can distinguish between singular points and other vertices of the graph, using equippings of edges adjacent to this vertices (if there are no such edges, then the vertex is necessarily a singular point). Thus we can recover the list of singular points from the graph.

2. Orbitally unstable paths A trajectory γ (it is called *path* in [1]) is called *ω -orbitally stable* at a point $M \in \gamma$ provided that $\forall \varepsilon > 0 \exists \delta > 0$ such that every trajectory $\gamma'(t)$, passing through δ -neighbourhood of M at $t = t_0$, remains in ε -neighbourhood of γ for $t > t_0$, *α -orbitally stable* trajectories are defined similarly. A trajectory is called *orbitally unstable* if it is not α -orbitally stable or not ω -orbitally stable at least at one point (see Definitions 14–17, §15). Note that in [1] this definition is applied to bounded semitrajectories only. In order to make it work on the sphere we have to use the spherical metric.

We claim that orbitally unstable trajectories are exactly limit cycles and separatrices. Indeed, limit cycles are orbitally unstable by Theorem 37 of §15. It is quite clear that the separatrices are orbitally unstable.

Conversely, assume that τ is an ω -unstable trajectory that is not a limit cycle. Theorem 40 of §15 shows that ω -limit set of τ consists of a single point. Theorem 38 of §15 tells that τ is a boundary curve of a hyperbolic sector.

Thus τ is a separatrix. Therefore we can recover the list of all orbitally unstable trajectories, using the equipplings of edges.

3. The remaining 6 types of singular elements deal with the boundary of a region. Only so-called *normal boundaries* are considered in [1] (see §16.2). The normal boundary is one that consists of finite number of arcs without contact and segments of trajectories (these segments are called *corner arcs*). These trajectories are not allowed to be separatrices or limit cycles of the field.

Thus the boundary of a P -field is not normal. Therefore we have to do the following: for a P -field v remove from the phase space the areas, bounded by boundary limit cycles and their corresponding cycles without contact. Then we get a new vector field v' with normal boundary. It is easy to see that P -fields v_1 and v_2 are orbitally topologically equivalent if and only if v'_1 and v'_2 are equivalent and the directions of rotation on the corresponding boundary cycles of v_1 and v_2 are the same. Thus we shall recover from graph the scheme of v' instead of that of v .

A.2 Scheme of a singular point

The local scheme of a monodromic singular point can be trivially recovered from the graph. The global scheme of a monodromic singular point is read from the corresponding cycle without contact (see Proposition A.2 of this paper).

According to Definition 23 of §19, a scheme of a characteristic singular point O is the list of **1)** all the separatrices of O ; **2)** all the separatrices of other singular points that tend to O ; **3)** elliptic sectors; **4)** the cyclic order of the above (recall that the boundary does not have corner arcs).

Elliptic sectors correspond to the loops of the small graph. They can be distinguished from other loops by their equippling. The separatrices are the other edges, adjacent to O . We can distinguish between separatrices of O and “foreign” separatrices, since we know for each germ of a separatrix whether it is a boundary trajectory of a hyperbolic sector (see §3.4 of this paper). The cyclic order is specified, since the graph is embedded into the sphere.

A.3 Limit continua

To comply with the terminology introduced in [1], we use the expressions *α -limit continuum* and *ω -limit continuum* as the synonyms for α -limit and ω -limit sets of trajectories.

A *cell* of a vector field is a connected component of the phase space after removal of all the singular elements. Consider the cell filled by closed

trajectories. This cell is doubly connected (see Theorem 50 of §16). A connected component of its boundary is called *0-limit continuum*.

One point limit continua are just attractors, repellers, and centers. Their schemes can be read from the graph (easy). Hence, we shall restrict ourselves to infinite limit continua.

Proposition A.1. *The infinite α -limit, ω -limit, and 0-limit continua that are not limit cycles are limit polycycles and vice versa, limit polycycles are limit continua. All the limit polycycles can be recovered from equipped graph.*

Proof. It follows from Lemma 3.2 of this paper that any infinite ω -limit continuum or α -limit continuum is a limit cycle or a limit polycycle. It can be proved similarly (using Theorem 70 of §23) that a 0-limit continuum is a 0-limit polycycle.

Conversely, it is clear that α -limit and ω -limit polycycles are limit continua. Consider a 0-limit polycycle X . We need to show that it is a 0-limit continuum. Consider a half-interval where the monodromy map is defined and the family of cycles, intersecting this half-interval. All these cycles belong to the same cell; X belongs to the boundary of this cell. Thus X is a part of 0-limit continuum. It follows from Theorem 70 of §23 and the uniqueness of continuation of a separatrix that X coincide with this 0-limit continuum.

It remains to show that the limit polycycles can be recovered from the graph. A trajectory ψ belongs to some limit polycycle if and only if this trajectory is equipped with 0, α , or ω on at least one of its sides. It remains to show that we can ascertain from the graph whether two separatrices belong to the same limit polycycle.

To this end we just need to check whether one separatrix is the continuation of the other (because a co-oriented separatrix has at most one continuation). This is the information we can get from the graph. Thus the whole limit polycycle can be recovered from the graph. \square

We know whether a continuum is α -limit, ω -limit, or 0-limit continuum from the equipping of any of its separatrices. The *global scheme* (see Definition 28, §25) of the continuum is the list of all the separatrices, tending to this continuum, with their cyclic order. This is read from the graph by looking at the corresponding cycle without contact, this is possible due to the following Proposition.

Proposition A.2. *The correspondence between limit polycycles (limit cycles, foci) and cycles without contact can be recovered from the graph.*

Proof. The large graph splits the sphere into the parts, we shall call them *faces*. There is a natural embedding of the large graph into the phase space. The part of the phase space, corresponding to the face under this embedding, is called *realization of a face*. Consider the case of polycycle (the other cases are similar). The proposition follows from the following claim: *A limit polycycle X and a cycle without contact Y such that Y belongs to the large graph, correspond to each other if and only if*

1. *They bound a face of the graph;*
2. *This face is on the positive side of X .*

The “only if” statement follows from the choice of the cycles without contact (see §3.3 of this paper).

Suppose X and Y bound a face and this face is on the positive side of X . Suppose, on the contrary, that Y corresponds to another limit polycycle, limit cycle, or focus X' . Since Y has been added to the graph, there is a separatrix τ such that τ intersects Y . Then the α -limit set or the ω -limit set of τ coincides with X' . Consider the first case (the second case is similar).

Since τ is a separatrix, its ω -limit set must be a single point O . But the ω -limit set of τ is contained in the closure of the realization of a face, bounded by X and Y , since Y is a cycle without contact. Thus O belongs to X (hence, X cannot be a limit cycle). But this contradicts the assumption that X is a limit polycycle and the face is on the positive side of X . \square

A.4 Boundary Scheme

The boundary (after removing neighbourhoods of boundary limit cycles) consists of cycles without contact, labelled by boundary limit cycles (notice that these cycles without contact may or may not belong to the large graph).

Let Y be a cycle without contact, corresponding to a boundary limit cycle X . Since the boundary contains no corner arcs, the global scheme of Y is the list specifying (see Definition 30 of §26): **1**) whether Y is an outer or inner boundary curve; **2**) whether Y is positive or negative cycle (i.e. whether X is an α -limit cycle or an ω -limit cycle); **3**) all singular paths, intersecting Y , enumerated in cyclic order.

Now **1**) and **2**) are read from the equipping of X . The list of all singular paths is recovered from vertices of the large graph (recall that if Y does not belong to the large graph, then there are no singular paths, intersecting Y).

A.5 Conjugate free continua

Two 0-limit continua are called *conjugate* if they bound a cell, filled with closed trajectories. Two cycles without contact are called conjugate if every trajectory that intersects the first one, intersects the second one as well. An α -limit continuum and an ω -limit continuum are called conjugate if the corresponding cycles without contact are conjugate (see §28 and §27.4).

It remains to show that we can recover all the pairs of conjugate free continua from the graph. (The continuum is called *free* if no separatrix intersects its cycle without contact. The conjugate continua are obviously free.)

Again, consider the faces of the large graph. It is enough to prove the following claim: *limit continua X and X' are conjugate if and only if they bound a face that is on the positive side of each continuum.* The “only if” part follows from Lemma 3a of §28.2.

Conversely, suppose X and X' bound a face, and this face is on the positive side of each of them. If the realization of this face is filled with closed phase curves, then X and X' are conjugate 0-limit continua. Otherwise let Y and Y' be cycles without contact, corresponding to X and X' respectively. We need to show that Y and Y' are conjugate cycles. On the contrary, assume that τ is a trajectory, τ intersects Y but does not intersect Y' . Since Y is a cycle without contact, τ intersects it just once. Thus either positive, or negative semitrajectory of τ belongs entirely to the region bounded by Y and Y' . But this region contains neither singular points nor limit cycles, this contradicts the Poincaré–Bendixon Theorem, which tells that α -limit (ω -limit) set of any trajectory contains either a singular point or a limit cycle.

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